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On transparent potentials: a Born approximation study

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Abstract. In the frame of the inverse scattering problem at fixed energy, we obtain a class of spherically symmetric potentials which are transparent in the Born approximation: all their Born phase-shifts vanish. This class consists of oscillating functions of the reduced radial variable. Amongst these functions, the Born approximation of the transparent potential of the Newton–Sabatier method is found. In the same class, we exhibit potentials for which the Born phase-shifts vanish at and after a certain L-wave (quasi-transparent potentials).

Very general features of potentials transparent in the Born approximation are then stated, and bounds are given for the exact scattering amplitudes corresponding to most of the potentials previously exhibited. These bounds, obtained at fixed energy and for large values of the angular momentum, are found to be independent of the energy.

1. Introduction

The problem of the uniqueness of the solution is fundamental in any study of an inverse scattering problem. In quantum scattering theory, it is well known that the resolution of the inverse problem at a fixed value of the angular momentum leads in general to a set of solutions (Gel'fand and Levitan 1951, Agranovich and Marchenko 1963). This set of solutions possesses a number of arbitrary parameters equal to the number of bound states which the potential admits. However, until now, no similar general result has been stated for the inverse problem at fixed energy. Certain special classes of potentials have been studied. On one hand, the uniqueness of the solution of an inverse problem at fixed energy was shown by Loeffel (1968) for Yukawa and finite-range potentials. On the other hand, the Newton–Sabatier method (Newton 1962, Sabatier 1966) is known to lead to a one-parameter family of equivalent potentials, amongst which a particular potential is chosen for physical reasons. As this last method concerns a limited class of functions, the observed lack of uniqueness may be thought to be an accidental one.

Outside the three quoted classes, a fundamental question remains unsolved: at a given energy, is the only knowledge of the phase-shifts corresponding to a potential sufficient, or not, to determine this potential uniquely? This paper does not claim to give an answer to this question. However, it allows a realisation of it by studying the problem in an approximative frame. This frame is chosen to be the Born approximation, for it linearises the phase-shifts as a function of the potential. Then, if all the Born phase-shifts of a potential vanish at a given energy, this potential may be added to any other potential without modifying in any way its set of Born phase-shifts at the energy considered. Such a potential is said to be transparent at this energy, and its addition to a

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given potential provides another potential equivalent to the first one. In our study, we exhibit a class of energy-dependent potentials which are transparent in the Born approximation at any energy. In the same class of functions, we obtain quasi-transparent potentials, i.e. potentials for which the Born phase-shifts vanish at and after a certain L-wave.

The search for such potentials was suggested to us by numerical tests that we performed in order to study carefully the results of the Newton–Sabatier method. So, in this paper, we begin in § 2 by a recall of some trials we carried out with the particular choice of an initial Gaussian potential. By the explicit computation of equivalent potentials, and of potentials generated by prematurely truncated sets of phase-shifts, this example shows the oscillating nature of the transparent and quasi-transparent potentials which occur in the method. In § 3, we tackle the question of a possible expression for a potential which is transparent or quasi-transparent in the Born approximation. We look for it inside a relatively wide class of functions of the reduced variable: the products of a power of the variable by a certain number of regular Bessel functions. One of the transparent potentials that we find of this form is nothing else than the Born approximation of the transparent potential of the Newton–Sabatier method. This result leads us to think that our potentials may be approximations for exact transparent potentials occurring in possible inverse methods at fixed energy. In § 4, use is made of the total Born scattering amplitude to derive very general features of potentials transparent in the Born approximation. Then, in order to give a more rigorous foundation to our work, we show in § 5 that the exact scattering amplitudes corresponding to most of our transparent potentials are bounded for large values of the angular momentum by a quantity which goes to zero when l goes to infinity. Finally we draw our conclusions.

2. The equivalent potentials of the Newton–Sabatier method

When the Newton–Sabatier method (Newton 1962, Sabatier 1966) is studied, the notion of equivalent potentials arises in a quite natural way. Let us recall that this method allows the construction of spherically symmetric potentials from only the knowledge of the set of phase-shifts $\{\delta_l\}$ at a given energy. Starting from such a set $\{\delta_l\}$, a one-parameter family of potentials may be generated. The potentials are different from each other at least in the asymptotic region, where they obey the following law (Sabatier 1966):

$$\mathcal{V}(x, \alpha) \xrightarrow[x\to\infty]{} -\frac{2}{\sqrt{\pi}}(\alpha-\beta)\frac{\cos(2x-\pi/4)}{x^{3/2}} + \mathrm{O}(x^{\epsilon-2}) \tag{1}$$

when δ_l goes to zero faster than $l^{-3-\epsilon}$ as l goes to infinity, i.e. for most situations occurring in nuclear physics. In equation (1), β is a number which can be computed from the initial $\{\delta_l\}$, and α is an arbitrary parameter. The variable is the reduced variable $x = kr$, r being the physical variable, and k the wavenumber associated with the energy.

Outside the asymptotic region, very little is known from theoretical considerations about the different potentials $\mathcal{V}(x, \alpha)$. Expanding them near the origin leads to

$$\mathcal{V}(x, \alpha) \xrightarrow[x\to 0]{} \frac{2c_0}{x} \tag{2}$$

where c_0 is a number which depends both on the initial set $\{\delta_l\}$ and on the choice of α. But, in the intermediate regions, where x is neither too small nor in the asymptotic zone, it seems to be difficult to estimate the behaviour of $\mathscr{V}(x, \alpha)$ by theoretical means.

So, let us consider a physical scattering of two particles of reduced mass μ, at a given energy $E_{CM} = \hbar^2 k^2/2\mu$. The knowledge of the phase-shifts $\{\delta_l\}$ produced by the interaction potential $V_i(r)$ leads to a family of potentials $V(r, \alpha) = E_{CM}\mathscr{V}(kr, \alpha)$. Each of them produces at the energy E_{CM} the same set of phase-shifts as $V_i(r)$. At this given energy, they are all equivalent to the initial potential, and equivalent to each other. However, the relative deviation between them is not theoretically known, and one cannot predict if one amongst the different $V(r, \alpha)$ is closer to $V_i(r)$ than the others. One may deduce from equation (2) a divergence of $V(r, \alpha)$ at the origin (except if $c_0 = 0$), and this behaviour is in most cases different from that of $V_i(r)$. However, it is well known that the experimental phase-shifts generally cannot provide information on the inner part of the potential; so this characteristic of $V(r, \alpha)$ does not seem very troublesome for eventual practical applications of the inverse method.

The asymptotic behaviour of $V(r, \alpha)$ is rather important from this last point of view. Equation (1) shows that the decrease of the different potentials generated by the method is generally slow. On the other hand, physical reasons usually lead us to imagine $V_i(r)$ as a short-ranged potential. In order to harmonise these two facts, Sabatier (1966) suggested the choice of the potential $V(r, \beta)$, which is the only one to decrease more rapidly than $r^{-3/2}$: this particular potential may then be convenient in nuclear physics, and reproduce satisfactorily the true potential $V_i(r)$. Let us recall that this choice unambiguously defines c_0, and therefore we know the behaviour of the potential near the origin.

In order to study more precisely the relative deviation of these equivalent potentials, numerical tests are required. A first series of tests was performed by Sabatier and Quyen Van Phu (1971, see also Chadan and Sabatier (1977)), who compared some possible potentials $V_i(r)$ with the corresponding $V(r, \beta)$ generated by the inverse method at different energies. We made another series of similar tests, in a more precise and complete way (Coudray 1977, 1979), so as to connect the characteristics of the potentials $V(r, \beta)$ with the internal parameters of $V_i(r)$. However, no potential $V(r, \alpha)$ had been computed for values of α different from β. To verify that the 'best' potential was indeed $V(r, \beta)$ we have explicitly calculated some of its equivalent potentials $V(r, \alpha)$.

We shall recall here two computational results which summarise very well most of our observations. They deal with the same initial Gaussian potential $V_i(r) = -V_0 \exp[-(r/\mu)^2]$, with $V_0 = 14$ MeV and $\mu = 3{\cdot}5$ fermi. We have calculated the set of phase-shifts produced by this potential in the scattering of a neutron of energy E_{lab} upon an α particle. In the centre-of-mass system, the energy E_{CM} is written $E_{CM} = \frac{4}{5}E_{lab}$, and if M is the nucleon mass, the reduced mass μ is equal to $\frac{4}{5}M$.

When the Newton–Sabatier method is applied, and $V(r, \beta)$ is computed, the result is very sensitive to the chosen energy. Figure 1 is a comparison between $V_i(r)$ and $V(r, \beta)$ for $E_{lab} = 10$, 30 and 400 MeV. For the lowest of these energies, the reproduction of the initial potential is relatively bad: the calculated potential oscillates around $V_i(r)$, and if this last potential was unknown, it would be difficult to deduce it from $V(r, \beta)$. At the energy of 30 MeV, the agreement is better: the only appreciable discrepancy between the initial potential and the calculated one takes place for values of r less than 2 fermi, and is easily related to the divergent expression of $V(r, \beta)$ near the origin (cf formula 2). For the third of the chosen energies, $E_{lab} = 400$ MeV, this discrepancy is

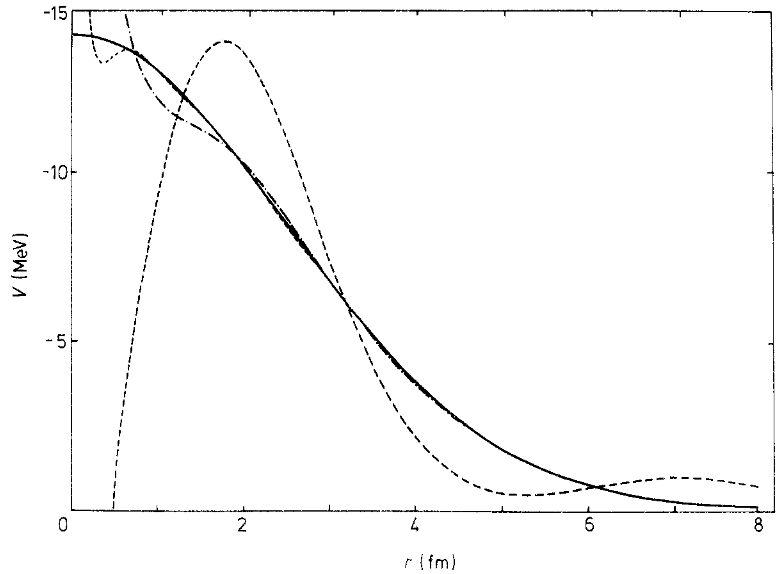

Figure 1. Results of the Newton–Sabatier method: $\alpha = \beta$. ——— $V_i(r) = -14\ e^{-(r/3\cdot5)^2}$

— — — — $E_{\text{lab}} = 10$ MeV, —·—·— $E_{\text{lab}} = 30$ MeV, - - - - - - $E_{\text{lab}} = 400$ MeV: calculated $V(r, \beta)$.

r is in fermi.

further reduced, and as soon as r becomes larger than 1 fermi, the agreement between $V_i(r)$ and $V(r, \beta)$ is excellent.

Now let us compute $V(r, \alpha)$ at this last energy of 400 MeV, for which $V(r, \beta)$ reproduces the initial potential fairly well. For $\alpha = 0\cdot9\beta$, we obtain the results shown in figure 2. The previous agreement is destroyed, and the computation leads to a strongly oscillating curve. Its oscillations are not limited to the asymptotic region, where theoretical considerations provided for them: they take place in the whole of the domain of variations of r. Their amplitude is relatively large; however, their average value seems to coincide with the initial potential. If other values of α are tested, they lead to similar oscillating curves (Coudray 1979).

These two examples illustrate the main characteristics of the potentials generated by the method. They show that the choice of $V(r, \beta)$ seems to be the best one in nuclear physics, and that, for a given initial potential $V_i(r)$, the higher energies give rise to the best fits. A more complete description of the potentials $V(r, \beta)$, including their dependence on the shape and the parameters of the initial potential, may be found in Coudray (1979). The results given here are generalised and, more precisely, the notion of 'critical energy' is introduced: a progressive decrease of the energy leads successively to a good reproduction of the initial potential, then to oscillations indicating the neighbourhood of this critical energy, and at last to the failure of the method. For the example chosen here, this critical energy may be approximately located around 10 MeV.

It is worthy of notice that, independently of our work, another calculation of $V(r, \beta)$ was performed with the help of the same techniques (Pelosi *et al* 1978); it was a direct attempt to explain in terms of potentials the experimental results of the π–N diffusion.

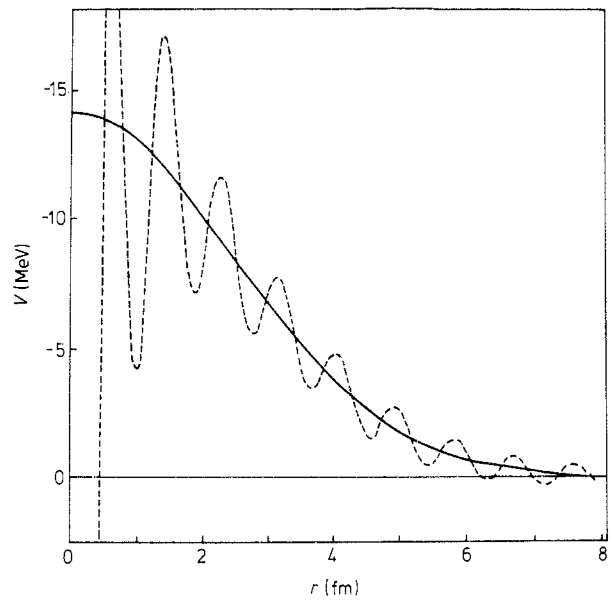

Figure 2. Results of the Newton–Sabatier method: $\alpha = 0{\cdot}9\beta$. ——— $V_i(r) = -14\,\mathrm{e}^{-(r/3{\cdot}5)^2}$, — — — — $E_{\mathrm{lab}} = 400$ MeV: calculated $V(r, \alpha)$. r is in fermi.

This computation led to strongly oscillating potentials. In this calculation, however, the origin of the observed oscillations is not easy to identify, which may be due to either the method itself, or the small number of experimental phase-shifts. Also, it was shown by Reignier (1979) that, in the class of potentials possessing a first absolute moment, any exact method of inversion with a finite number of phase-shifts leads always to potentials which own oscillating tails.

The influence of the number of phase-shifts introduced into the computation can be numerically studied in the frame of the Newton–Sabatier method. Choosing again our Gaussian potential, at the energy $E_{\mathrm{lab}} = 400$ MeV, we were sure that the eventual oscillations could not be due to the method itself: at this energy, when the number N of phase-shifts is sufficiently large ($L = 30$), the computed potential $V(r, \beta)$, as shown in figure 2, possesses no visible oscillation. We successively reduced this number L to the values 20, 15, 10, 5, 3 and 1, obviously, always choosing the first phase-shifts. Our results, shown in figure 3, indicate that in any case the truncation of the series of phase-shifts leads to oscillations, situated within the whole domain of variations of r. When the truncation is very premature ($L = 1, 3, 5$), most of the oscillations seem to admit the zero potential as an average value. When supplementary phase-shifts are added ($L = 10, 15, 20$), the oscillations centre progressively on the initial potential, until L becomes large enough ($L \simeq 30$) to lead to a non-oscillating curve.

All the potentials drawn in figure 3 are quasi-transparent. Their first L phase-shifts are identical to those of $V_i(r)$. They all oscillate, and the oscillations obtained by Pelosi *et al* (1978) may be due to the same phenomena.

The different numerical examples we have recalled here are in agreement with previous theoretical results which provided for oscillations in the asymptotic region. However, we observe oscillations on the whole domain of definition of the potential. Is it then possible to understand the origin of these oscillations? In other words, can we

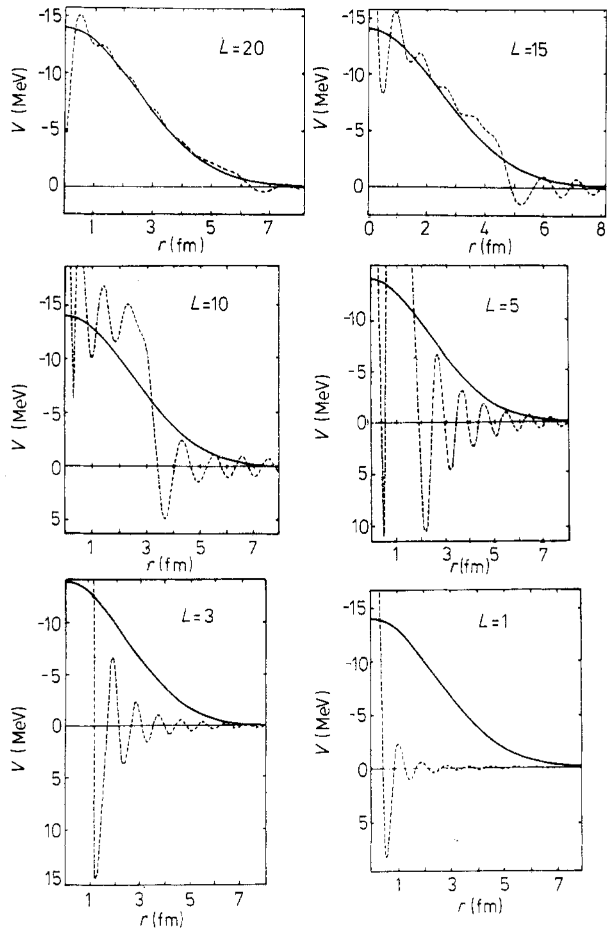

Figure 3. Influence of the number of phase-shifts in the Newton–Sabatier method. ——— $V_i(r) = -14\,e^{-(r/3.5)^2}$, – – – – $E_{lab} = 400$ MeV: calculated potential $V(r, \beta)$ with the first L phase-shifts. r is in fermi.

write a possible expression for a potential equivalent to another one, or for a quasi-transparent potential? In what follows, we try to give an approximate answer to this question.

3. Obtainment of potentials transparent or quasi-transparent in the Born approximation

3.1. Introduction of the Born approximation

A particularly interesting approximative study of the previous problem may be done in the frame of the Born approximation. Within this approximation, the phase-shift becomes a linear function of the potential:

$$\sin \delta_{l_B} = -\frac{2\mu k}{\hbar^2} \int_0^\infty j_l^2(kr) V(r) r^2 \, dr. \tag{3}$$

In this formula, $j_l(r)$ is the regular spherical Bessel function of order l. The linearity of $\sin\delta_{l_B}$ implies that the difference between two equivalent potentials at a given energy is a potential for which all phase-shifts vanish at this energy, a transparent potential by definition. So the search for equivalent potentials reduces to the search for transparent potentials. If we are able to exhibit transparent potentials, the addition of one of them to any potential will provide a potential equivalent to this last one.

How can a transparent potential be defined in the Born approximation? For such a potential $V_{\mathrm{T}}(r)$ every phase-shift vanishes, so that

$$\int_0^\infty j_l^2(kr)V_{\mathrm{T}}(r)r^2\,\mathrm{d}r=0 \qquad \text{for any integer } l\geqslant 0 \tag{4}$$

or, after introducing the reduced variable $x=kr$, the reduced potential $\mathscr{V}_{\mathrm{T}}(x)=(2\mu/\hbar^2k^2)V_{\mathrm{T}}(x/k)$ and the ordinary Bessel functions $\mathrm{J}_{l+\frac{1}{2}}(x)=(2x/\pi)^{1/2}j_l(x)$:

$$\int_0^\infty \mathrm{J}_{l+\frac{1}{2}}^2(x)\mathscr{V}_{\mathrm{T}}(x)x\,\mathrm{d}x=0 \qquad \text{for any integer } l\geqslant 0. \tag{5}$$

So we have to look for a function $\mathscr{V}_{\mathrm{T}}(x)$ orthogonal on $(0,+\infty)$ to every function $x\mathrm{J}_{l+\frac{1}{2}}^2(x)$.

3.2. Potentials transparent in the Born approximation

In order to derive a possible expression for $\mathscr{V}_{\mathrm{T}}(x)$, we worked in steps. First we looked for $\mathscr{V}_{\mathrm{T}}(x)$ amongst the functions $\mathscr{V}_{\mathrm{T}}(x)$ such that $x^\gamma\mathscr{V}_{\mathrm{T}}(x)$ is square-integrable (with γ real). Although we were then working in a reduced class, we enjoyed the advantage that these functions could be decomposed with respect to the set of eigenfunctions of differential self-adjoint operators (Dunford and Schwartz 1964). With the help of this last technique, we were led to the following expression for $\mathscr{V}_{\mathrm{T}}(x)$ (Coudray 1979):

$$\mathscr{V}_{\mathrm{T}}(x)=[\mathrm{J}_0(\kappa x)]/x \qquad \kappa\geqslant 2 \tag{6}$$

which corresponds to a very narrow class of functions. However, we are now in possession of an analytic expression for $\mathscr{V}_{\mathrm{T}}(x)$, and it is easy to generalise it. A first extension may be the class of functions which are products of a power of x by a Bessel function. The introduction of such functions into equation (5) provides

$$\mathscr{V}_{\mathrm{T}}(x)=x^{\nu-1}\mathrm{J}_\nu(cx) \tag{7a}$$

$$-\tfrac{1}{2}<\nu<\tfrac{1}{2} \tag{7b}$$

$$c>2. \tag{7c}$$

But inequality (7*b*)—which is obtained by imposing on $\mathscr{V}_{\mathrm{T}}(x)$ the usual conditions for the applicability of scattering theory—cannot allow an asymptotic decrease of $\mathscr{V}_{\mathrm{T}}(x)$ faster than $x^{-2+\epsilon}$, $\epsilon>0$. Also, a recent result of Reignier (1979) shows that a potential transparent in the Born approximation may decrease more rapidly than any prescribed negative power of x. This contradiction may be removed if we extend a second time the class of the required functions, by considering products of powers of x by several Bessel functions:

$$\mathscr{V}_{\mathrm{T}}(x)=x^sJ_{\nu_1}(c_1x)\mathrm{J}_{\nu_2}(c_2x)\ldots\mathrm{J}_{\nu_m}(c_mx). \tag{8}$$

This last expression allows an arbitrary asymptotic power decrease of $\mathscr{V}_{\mathrm{T}}(x)$, each Bessel function reducing it by an amount of $\frac{1}{2}$.

Let us discuss the possibility for $\mathscr{V}_{\mathrm{T}}(x)$ as defined by equation (8) to be transparent in the Born approximation—this discussion will include the two previous particular cases. We restrict our discussion to real potentials, i.e. to real parameters s, ν_i and c_i. The condition (5) reads

$$I_l=\int_0^\infty \mathrm{d}x\, x^{s+1}[\mathrm{J}_{l+\frac{1}{2}}(x)]^2\mathrm{J}_{\nu_1}(c_1x)\mathrm{J}_{\nu_2}(c_2x)\ldots\mathrm{J}_{\nu_m}(c_mx)=0 \qquad \text{for any integer } l\geqslant 0. \tag{9}$$

The convergence of I_l must be ensured, both in the vicinity of the origin, and for large values of x. This leads to the double condition

$$s+3+\nu_1+\nu_2+\ldots+\nu_m>0 \tag{10}$$

$$s<m/2. \tag{11}$$

The value of I_l is known: the simple change of functions (Bateman 1953 I)

$$\mathrm{J}_p(x)=(x/2)^p{}_0F_1(-;p+1;-x^2/4) \tag{12}$$

converts I_l into a tabulated integral (Exton 1978):

$$\begin{aligned}I_l=&\frac{(c_1)^{\nu_1}(c_2)^{\nu_2}\ldots(c_m)^{\nu_m}}{2^{2l+1+\nu_1+\ldots+\nu_m}}\frac{(4c_m)^{-2l-3-s-\nu_1-\ldots-\nu_m}}{\Gamma^2(l+\frac{3}{2})\Gamma(\nu_1+1)\ldots\Gamma(\nu_m+1)}\\&\times\frac{\Gamma(\nu_m+1)\Gamma(2l+3+s+\nu_1+\nu_2+\ldots+\nu_m)}{\Gamma[l+2+\frac{1}{2}(s+\nu_1+\nu_2+\ldots+\nu_m)]\Gamma[-\frac{1}{2}-l+\frac{1}{2}(\nu_m-s-\nu_1-\nu_2-\ldots-\nu_{m-1})]}\\&\times F_c^{(m+1)}[l+\tfrac{3}{2}+\tfrac{1}{2}(s+\nu_1+\nu_2+\ldots+\nu_m),\ l+\tfrac{3}{2}\\&+\tfrac{1}{2}(s-\nu_m+\nu_1+\nu_2+\ldots+\nu_{m-1});\ l+\tfrac{3}{2},\ l+\tfrac{3}{2},\ \nu_1+1,\ldots\\&\ldots,\nu_{m-1}+1;\ 1/c_m^2,\ 1/c_m^2,\ (c_1/c_m)^2,\ldots,(c_{m-1}/c_m)^2]\end{aligned} \tag{13}$$

where $F_c^{(m+1)}$ is the Lauricella function of $(m+1)$ variables, a function which reduces to F_4 if $m=1$†. As soon as this function may be defined as a multiple series, it cannot become singular, and the cancellation of I_l for any integer $l\geqslant 0$ may occur if the denominator grows towards infinity. This happens whenever the relation

$$-\tfrac{1}{2}-l-\tfrac{1}{2}(\nu_0-s)=-n$$

or

$$s=\nu_0-1-2l+2n \tag{14}$$

is fulfilled. In the equations, ν_0 is defined as

$$\nu_0=\nu_m-(\nu_1+\nu_2+\ldots+\nu_{m-1}) \tag{15}$$

and n is any integer $\geqslant 0$. Given ν_0, and given a value of n, it is always possible to choose s by imposing $l=0$ in equation (14). This last equation is then automatically fulfilled for any integer $l>0$. So, given ν_0, it is easy to annihilate I_l for any integer $l\geqslant 0$ by setting

$$s=\nu_0-1+2n, \tag{16}$$

n being any integer $\geqslant 0$.

† The quoted reference provides $(c_m/4)^{-2l-3-s-\nu_1-\ldots-\nu_m}$ instead of $(4c_m)^{-2l-3-s-\nu_1-\ldots-\nu_m}$. However, as the resulting formula does not reduce for $m=0$ and $m=1$ to the well known formulae (see for instance Bailey (1934) and Bateman (1953 II)), we have slightly modified it. Our results are independent of this modification.

If positive parameters c_i are assumed†, $F_c^{(m+1)}$ may be defined as a multiple series since the following inequality is fulfilled:

$$2+c_1+c_2+\ldots+c_{m-1}<c_m. \tag{17}$$

Then equations (15) and (16), together with conditions (10), (11) and (17), define a class of functions $\mathscr{V}_{\mathrm{T}}(x)$:

$$\mathscr{V}_{\mathrm{T}}(x)=x^{\nu_0+2n-1}\mathrm{J}_{\nu_1}(c_1x)\mathrm{J}_{\nu_2}(c_2x)\ldots\mathrm{J}_{\nu_m}(c_mx) \tag{18}$$

for which every integral I_l vanishes. But we require more on $\mathscr{V}_{\mathrm{T}}(x)$: we want to be able to apply scattering theory to them, and, amongst the previous class, this leads us to select functions such that

$$|\mathscr{V}_{\mathrm{T}}(x)|<Ax^{-1-\eta} \qquad \text{when } x\to\infty$$
$$|\mathscr{V}_{\mathrm{T}}(x)|<Bx^{-2+\eta'} \qquad \text{when } x\to 0$$

where η and η' are positive arbitrary constants. These two conditions limit the domain of variations of the parameters n and ν_i according to the inequality

$$-(n+\tfrac{1}{2})<\nu_m<(\nu_1+\nu_2+\ldots+\nu_{m-1})+m/2-2n \tag{19}$$

which involves conditions (10) and (11). So, from now on, our potentials will be defined by equations (15) and (18), and submitted to conditions (17) and (19).

A graphical study of inequality (19) may be done: see figure 4. Let us suppose the parameters ν_i to be known, and let us look for the possible values for n. In figure 4, all these values are the abscissae of points contained either inside the triangle ABC, or on the segment AC. The straight line BC possesses a constant slope, but its position is related to the value of $q=m/2+\nu_1+\ldots+\nu_{m-1}$. The abscissa of B, $n_{\mathrm{B}}=q+\frac{1}{2}$, corresponds to the upper bound for n, a bound which cannot be reached. So n belongs to the set $\{0, 1, 2\ldots N\}$, where N is the largest integer less than n_{B}.

However, the double inequality (19) does not involve very restrictive conditions on $\mathscr{V}_{\mathrm{T}}(x)$. Indeed, near the origin, the potential behaves like $x^{2(n+\nu_m)-1}$, i.e. may have any prescribed behaviour compatible with quantum scattering theory. On the other hand, the potential becomes asymptotically

$$\mathscr{V}_{\mathrm{T}}(x)\xrightarrow[x\to\infty]{}\frac{\cos(c_1x-\frac{1}{2}\nu_1\pi-\frac{1}{4}\pi)\cos(c_2x-\frac{1}{2}\nu_2\pi-\frac{1}{4}\pi)\ldots\cos(c_mx-\frac{1}{2}\nu_m\pi-\frac{1}{4}\pi)}{x^{1-2n+m/2-\nu_0}} \tag{20}$$

and it may decrease more rapidly than any given negative power of x: even when the sum $(n+\nu_m)$ is given, the quantity $1+q-n=1+q-(n+\nu_m)+\nu_m$ remaining in the exponent of x may be chosen arbitrarily large. So every decreasing power of x compatible with scattering theory may asymptotically be reached by some $\mathscr{V}_{\mathrm{T}}(x)$.

Obviously, the expressions (6) and (7) correspond to particular cases of the general potential (18), and may be derived with the same techniques. However, as said before, we were led to the expression (6) by another method, and we were able to show that the limit value $\kappa=2$ could be included in the range of the parameter κ (Coudray 1979). It was therefore interesting to look for the possibility for the parameter c of equation (7)

† According to the formula (Bateman 1953 III)

$$\mathrm{J}_\nu(\mathrm{e}^{\mathrm{i}m\pi}x)=\mathrm{e}^{\mathrm{i}m\nu\pi}\mathrm{J}_\nu(x)$$

the choice of some parameters $-c_i$ instead of $+c_i$ leads to the same potential $\mathscr{V}_{\mathrm{T}}(x)$, to within a multiplicative phase.

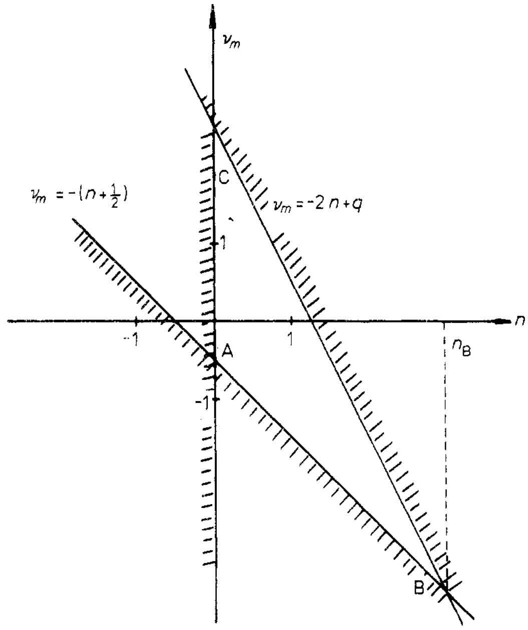

Figure 4. Possible values of n. $q = \frac{1}{2}m + (\nu_1 + \nu_2 + \ldots + \nu_{m-1})$.

to be equal to 2, and, more generally, for c_m to reach the sum $2 + c_1 + \ldots + c_{m-1}$. In the Appendix, use is made of Lebesgue's theorem to show that this limit value may be included in the range of c_m, so that inequality (17) is to be replaced by

$$2 + c_1 + c_2 + \ldots + c_{m-1} \leqslant c_m. \tag{21}$$

In particular, if $m = 1$, the value $c = 2$ is allowed.

3.3. *Link with the Newton–Sabatier method*

Let us now come back to our starting point, i.e. to the Newton–Sabatier method. It has been shown that an exact transparent potential is generated by this method, that it is unique (Redmond 1964), and that it behaves at infinity like $[\cos(2x - \pi/4)]/x^{3/2}$ (cf formula (1)) (Sabatier 1966). If we compare this asymptotic behaviour with equation (20), we observe that it may be that of $\mathcal{V}_{\mathrm{T}}^0(x) = [\mathrm{J}_0(2x)]/x$, one of the transparent potentials we have defined, corresponding to $m = 1$, $n = \nu = 0$ and $c = 2$. Furthermore, $\mathcal{V}_{\mathrm{T}}^0(x)$ has the same behaviour in the vicinity of the origin as those potentials obtained via the Newton–Sabatier method. These two similarities lead us to think that $\mathcal{V}_{\mathrm{T}}^0(x)$ may be the Born approximation of the exact transparent potential of this method.

In order to verify this assumption, we shall use the numerical results shown in figure 2. Let us suppose the Born approximation to be valid. Then the property of linearity of the phase-shifts implies that the computed potential may be the sum of the initial potential $V_i(r)$ and, if our assumption concerning $\mathcal{V}_{\mathrm{T}}^0$ is valid, of any multiple of $V_{\mathrm{T}}^0(r) = E_{\mathrm{CM}}\mathcal{V}_{\mathrm{T}}^0(kr)$. So

$$V^{\mathrm{B}}(r, \alpha) = E_{\mathrm{CM}}\mathcal{V}(kr, \alpha) = V_i(r) + KE_{\mathrm{CM}}\mathcal{V}_{\mathrm{T}}^0(kr). \tag{22}$$

As the Gaussian potential decreases more rapidly than $\mathscr{V}_T^0(kr)$, asymptotically $V^B(r, \alpha)$ reduces to the multiple of the transparent potential and

$$V^B(r, \alpha) \xrightarrow[r\to\infty]{} KE_{CM} \frac{\cos(2kr - \pi/4)}{(kr)^{3/2}} + \ldots . \tag{23}$$

But this is to be identified with the asymptotic behaviour of $V(r, \alpha)$, as given by equation (1). Now K is defined unambiguously as

$$K = -2(\alpha - \beta). \tag{24}$$

In figure 5 are drawn the potential $V(r, \alpha)$ obtained previously from the set of phase-shifts $\{\delta_l\}$ corresponding to $V_i(r)$, and the potential $V^B(r, \alpha)$ defined by equations (22) and (24). A very good agreement holds between these two potentials, particularly near the origin and in the tail region. The slight discrepancy observed for the intermediate values of r may be due to the fact that the true transparent potential involved in the Newton–Sabatier method is probably not exactly equal to its Born approximation.

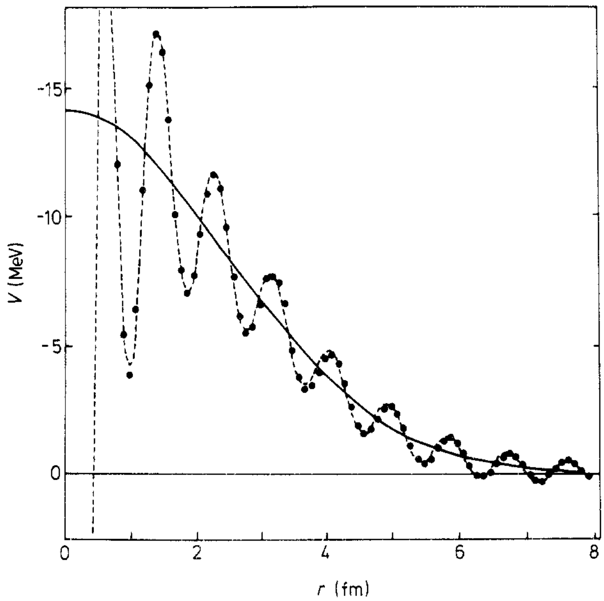

Figure 5. Comparison between $V(r, \alpha)$ and $V^B(r, \alpha) = V_i(r) - 2(\alpha - \beta)E_{CM}(J_0(2kr)/kr)$. ——— $V_i(r) = -14\,e^{-(r/3\cdot5)^2}$, — — — $V(r, \alpha)$, • • • $V^B(r, \alpha)$. r is in fermi.

Nevertheless, we can conclude that our class of transparent potentials contains the Born approximation of the Newton–Sabatier transparent potential. This last result allows the best understanding of reasons for the non-uniqueness of the solution of the studied method. Furthermore, it allows us to predict analogous lacks of uniqueness in any possible inverse method based only on the knowledge of the phase-shifts of a potential at a given energy.

3.4. Potentials quasi-transparent in the Born approximation

Techniques similar to those of § 3.2 may be used to determine a class of quasi-transparent potentials belonging to the same set of functions. For such potentials, all phase-shifts vanish at and after some wave L. So, in the Born approximation, we must impose the cancellation of I_l for any integer $l \geqslant L$. This requires the fulfilment of the condition

$$\delta = \nu_0 - 2L - 1, \tag{25}$$

and the corresponding quasi-transparent potentials can be written

$$\mathscr{V}_{\mathrm{TL}} = x^{\nu_0 - 2L - 1} \mathrm{J}_{\nu_1}(c_1 x) \mathrm{J}_{\nu_2}(c_2 x) \ldots \mathrm{J}_{\nu_m}(c_m x) \tag{26}$$

with parameters submitted to the following conditions:

$$\begin{aligned} &L - \tfrac{1}{2} < \nu_m < (\nu_1 + \ldots + \nu_{m-1}) + m/2 + 2L \\ &2 + c_1 + \ldots + c_{m-1} \leqslant c_m. \end{aligned} \tag{27}$$

These last inequalities may be deduced in a similar manner to the conditions concerning the transparent potentials. Like them, they are not very restrictive, and allow $\mathscr{V}_{\mathrm{TL}}(x)$ an arbitrary asymptotic power decrease.

To conclude this section, we want to emphasise the fact that all the potentials we have obtained are energy-dependent: the variable x is the reduced variable and the formula

$$V_{\mathrm{TL}}^{\mathrm{T}}(r) = E_{\mathrm{CM}} \mathscr{V}_{\mathrm{TL}}^{\mathrm{T}}(kr) \tag{28}$$

shows this dependence.

4. Use of the total Born scattering amplitude

Some general characteristics of the potentials transparent in the Born approximation may be deduced from the introduction of the total Born scattering amplitude. With its help, the condition of transparency in the Born approximation may be written (Sabatier 1973, Reignier 1979) in a form different from equation (5). Let us write the total Born scattering amplitude corresponding to a potential $V(r)$ at the energy $E_{\mathrm{CM}} = \hbar^2 k^2 / 2\mu$:

$$h_{\mathrm{B}}(q) = -\frac{2\mu}{\hbar^2 q} \int_0^\infty r \sin qr \, V(r) \, \mathrm{d}r. \tag{29}$$

This is the sine Fourier transform of the function $rV(r)$. In this last formula, the variable q is the length of the momentum transfer $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$, $\boldsymbol{k}$ being the wavevector of the incident particle, and $\boldsymbol{k}'$ a vector of the same length, but parallel to the direction of observation. A potential will be transparent if $h_{\mathrm{B}}(q)$ vanishes for every physical value of q, i.e. for $q \in [0, 2k]$.

This definition of transparency allows us to derive very general features of the transparent potentials. Let us state them.

Property 1. Every potential transparent in the Born approximation at a given energy is transparent in the same approximation at any lower energy.

This result is obvious. If $h(q) = 0$ for $0 \leqslant q \leqslant 2k$, then $h(q) = 0$ for $0 \leqslant q \leqslant 2k_1$ for any $k_1 < k$.

Property 2. If a potential $V_T(r)$ is transparent in the Born approximation at a given energy E, and if γ is a real positive number, the potential $V_T(\gamma r)$ is transparent in the same approximation at the energy $\gamma^2 E$.

This property is easily shown. The Born scattering amplitude corresponding to $V_\gamma(r) = V_T(\gamma r)$, as defined in property 2, may be written

$$h_{\gamma,B}(q) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty \sin qr V_T(\gamma r) r \, dr$$

$$= -\frac{2\mu}{\hbar^2 q\gamma^2}\int_0^\infty \sin\frac{qr'}{\gamma} V_T(r') r' \, dr'$$

$$h_{\gamma,B}(q) = \frac{1}{\gamma^2} h_B\left(\frac{q}{\gamma}\right)$$

where $h_B(q)$ is the Born scattering amplitude corresponding to $V_T(r)$. As this last quantity vanishes for $q \in [0, 2k]$, $h_{\gamma,B}(q)$ vanishes for $q \in [0, 2k\gamma]$, so $V_\gamma(r)$ is transparent at the energy $\gamma^2 E$.

Now let us suppose $\gamma > 1$; $V_\gamma(r)$ is transparent at the energy $E_\gamma = \gamma^2 E$, $E_\gamma > E$, so property 1 implies that $V_\gamma(r)$ is still transparent at the energy E. On the other hand, for $\gamma < 1$, $V_\gamma(r)$ is generally not transparent at the energy E. Consequently, we may write the important following property.

Property 3. Starting from a given potential $V_T(r)$ transparent in the Born approximation at the energy E, one may generate a class of potentials possessing the same property by the substitution of γr for r in the expression of $V_T(r)$, γ being any real number greater than one.

This last property may be used to explain the origin of inequality (21). This inequality, obtained by other means, indicates that the parameter c_m is limited by a lower bound. Let us apply property 3 to the potential $V_T(r) = E_{CM}\mathcal{V}_T(kr)$, where $\mathcal{V}_T(x)$ is given by (18), with $c_m = 2 + c_1 + c_2 + \ldots c_{m-1}$. $V_T(r)$ is transparent at the energy $E_{CM} = \hbar^2 k^2/2\mu$. For any $\gamma > 1$, the potential $V_T(\gamma r)$ is transparent at the same energy. Now its parameters are $\gamma c_1, \gamma c_2, \ldots, \gamma c_m$, such that

$$\gamma c_m = 2\gamma + \gamma c_1 + \ldots + \gamma c_{m-1} \qquad \text{so} \qquad \gamma c_m > 2 + \gamma c_1 + \ldots + \gamma c_{m-1}.$$

Therefore, for every potential $V_T(\gamma r)$, inequality (21) holds. On the other hand, for $\gamma < 1$, $V_T(\gamma r)$ is generally not transparent at the energy E_{CM}, and the sense of the inequality cannot be reversed.

5. Bounds for the exact scattering amplitude of our transparent potentials

We are now in possession of a relatively wide class of potentials transparent in the Born approximation. However, one last question remains unsolved. What is the validity of our results? In other words, is the Born approximation a 'good' approximation when the energy is fixed? In this section, we shall give elements for an answer to this question.

The validity of the Born approximation at fixed energy according to the values of the angular momentum was studied by Martin (1964) for potentials $V(r)$ such that

(i) $V(r)$ possesses an asymptotic constant sign;

(ii) the integral $\int_0^\infty r|V(r)| \, dr$ does exist.

Although the first of these conditions is never fulfilled by our potentials $V_T(r)$, and the second one is generally violated, it is still possible to adapt a large part of Martin's work. In order to do this, we define the exact scattering amplitude corresponding to a transparent potential $V_T(r)$, and its Born approximation:

$$f_l = -\frac{1}{k}\int_0^\infty \mathrm{d}r\,\hat{j}_l(kr)u_l(r)U_T(r) \tag{30a}$$

$$f_l^{\mathrm{B}} = -\frac{1}{k}\int_0^\infty \mathrm{d}r\,\hat{j}_l^2(kr)U_T(r) \tag{30b}$$

where $\hat{j}_l(kr) = krj_l(kr)$, $U_T(r) = (2\mu/\hbar^2)$, $V_T(r) = k^2\mathscr{V}_T(kr)$ and $u_l(r)$ is the regular solution of the radial Schrödinger equation for the lth partial wave. The validity of the Born approximation is proved if, for any given $\epsilon > 0$, the quantity $R_l = |f_l - f_l^{\mathrm{B}}|/|f_l^{\mathrm{B}}|$ is less than ϵ as soon as l is greater than some $l(\epsilon)$. For our transparent potentials $V_T(r)$, f_l^{B} vanishes always identically, and we cannot verify such an assertion. However, R_l may be written

$$R_l = \frac{|f_l - f_l^{\mathrm{B}}|}{|f_l^{\mathrm{B}'}|}\frac{|f_l^{\mathrm{B}'}|}{|f_l^{\mathrm{B}}|} = R_{l_1}R_{l_2} \tag{31}$$

after introducing the Born scattering amplitude corresponding to $|V_T(r)|$,

$$f_l^{\mathrm{B}'} = -\frac{1}{k}\int_0^\infty \mathrm{d}r\,\hat{j}_l^2(kr)|U_T(r)|, \tag{32}$$

and the study of R_l may be done in two steps (Martin 1964). With our potentials, the quantity R_{l_1} may be studied and bounded. On the other hand, R_{l_2} cannot be defined. In order to improve somewhat the bound on R_{l_1}, it is possible to introduce the difference $\||f_l^{\mathrm{B}'}| - |f_l^{\mathrm{B}}|\|$, which reduces to $|f_l^{\mathrm{B}'}|$, and to bound it independently of R_{l_1}. This double study allows us to obtain a bound for $|f_l|$ which decreases like a negative power of l, and which, furthermore, does not depend on the energy.

5.1. *Study of* R_{l_1}

We shall adopt the techniques of Martin (1964). We first introduce

$$|f_l - f_l^{\mathrm{B}}| \leqslant \frac{2\mu}{\hbar^2 k}\left|\int_0^\infty [u_l(r) - \hat{j}_l(kr)]\hat{j}_l(kr)\,\mathrm{d}r\right|. \tag{33}$$

We know that $u_l(r)$ is a solution of the integral equation

$$u_l(r) = \hat{j}_l(kr) - \frac{1}{k}\int_0^\infty \mathrm{d}r'\,\hat{j}_l(kr_<)\hat{h}_l^{(1)}(kr_>)U_T(r')u_l(r'). \tag{34}$$

In this equation, $\hat{h}_l^1(kr) = kr\,h_l^1(kr)$, $h_l^{(1)}$ being the spherical Hankel function of the first kind, and $kr_<$ and $kr_>$ denote the smaller and greater respectively of kr and kr'. Let us recall that $U_T(r) = k^2\mathscr{V}_T(kr)$ is energy dependent.

So the following bound holds:

$$|u_l(r)| \leqslant |\hat{j}_l(kr)| + \frac{1}{k}\int_0^\infty \mathrm{d}r'\,|\hat{j}_l(kr_<)\hat{h}_l^{(1)}(kr_>)||U_T(r')||u_l(r')| \tag{35}$$

and in order to bound the quantity $|\hat{j}_l(kr_<)\hat{h}_l^{(1)}(kr_>)|$, we use the two inequalities (Martin 1964)

$$\left.\begin{aligned} &|\hat{j}_l(r)\hat{h}_l^{(1)}(r')|\leqslant c(rr')^{1/2}\left(\frac{\pi}{2l+1}\right)^{1/2} \quad (36a)\\ &|\hat{j}_l(r)\hat{h}_l^{(1)}(r')|\leqslant c_1' l^{1/3}+c_1''\dagger \quad (36b)\end{aligned}\right\}\quad \text{for } r\leqslant r';\ c, c_1', c_1''>0.$$

Let us introduce $a\in[0, 1]$; we deduce from the last inequalities

$$|\hat{j}_l(r)\hat{h}_l^{(1)}(r')|\leqslant c'(rr')^{a/2}(2l+1)^{-a/2}(c_1' l^{1/3}+c_1'')^{1-a} \qquad c'>0. \tag{37}$$

Following Martin, we substitute this bound in equation (35). Then we multiply this equation by $r^{a/2}U_{\mathrm{T}}(r)$, and integrate over r. We obtain

$$\begin{aligned}\int_0^\infty |u_l(r)|r^{a/2}|U_{\mathrm{T}}(r)|\,\mathrm{d}r &\leqslant \int_0^\infty |\hat{j}_l(kr)|r^{a/2}|U_{\mathrm{T}}(r)|\,\mathrm{d}r\\ &\quad + c'(c_1' l^{1/3}+c_1'')^{1-a}(2l+1)^{-a/2}k^{a-1}\int_0^\infty r^a|U_{\mathrm{T}}(r)|\,\mathrm{d}r\\ &\quad \times\int_0^\infty \mathrm{d}r'\, r'^{a/2}|U_{\mathrm{T}}(r')|\,|u_l(r')|\end{aligned} \tag{38}$$

as long as every integral involved converges. Let us first consider $A(a, k)$:

$$\begin{aligned}A(a, k) &= \int_0^\infty r^a|U_{\mathrm{T}}(r)|\,\mathrm{d}r\\ &= k^2\int_0^\infty r^a|\mathcal{V}_{\mathrm{T}}(kr)|\,\mathrm{d}r\\ &= k^2k^{-1-a}\int_0^\infty x^a|\mathcal{V}_{\mathrm{T}}(x)|\,\mathrm{d}x = k^{1-a}A(a),\end{aligned} \tag{39}$$

an expression which shows that $k^{a-1}A(a, k)=A(a)$ does not depend on k. The integral $A(a, k)$ exists if a is chosen such that (see equations (15) and (18))

$$a<m/2-\nu_0-2n \tag{40a}$$

$$a>-2(\nu_m+n). \tag{40b}$$

If these two conditions are not compatible (compare with inequality (19)), we shall not be able to give any bound on $|f_l|$. Let us suppose that this is not the case, and that a value of a does exist which fulfils these two inequalities. Then let us introduce

$$I_l=\int_0^\infty r^{a/2}|U_{\mathrm{T}}(r)|\,|u_l(r)|\,\mathrm{d}r. \tag{41}$$

An easy estimation shows that I_l converges too. We can write

$$I_l[1-c'(c_1' l^{1/3}+c_1'')^{1-a}(2l+1)^{-a/2}A(a)]\leqslant\int_0^\infty |\hat{j}_l(kr)|r^{a/2}|U_{\mathrm{T}}(r)|\,\mathrm{d}r$$

or

$$I_l[1-X_a(l)]\leqslant\int_0^\infty |\hat{j}_l(kr)|r^{a/2}|U_{\mathrm{T}}(r)|\,\mathrm{d}r \tag{42}$$

† This inequality is not demonstrated in Martin's course. We have tested it numerically (see Coudray 1979).

where we have set

$$X_a(l) = c'(c_1' l^{1/3} + c_1'')^{1-a}(2l+1)^{-a/2} A(a). \tag{43}$$

For large values of l, we have

$$(c_1' l^{1/3} + c_1'')^{1-a}(2l+1)^{-a/2} = \mathrm{O}[l^{1/3-5a/6}]$$

and this quantity is a decreasing function of l as soon as

$$a > \tfrac{2}{5}. \tag{44}$$

We shall hereafter impose this condition on a, i.e. limit the domain of variations of a to the interval $]\frac{2}{5}, +1]$. If in this domain a value of a allows the existence of $A(a)$, then there exists some l_m such that

(i) $X_a(l_m) \leqslant 1$,

(ii) for $l > l_m$, $X_a(l_m)$ is a strictly decreasing function of l. It is noteworthy that the value of l_m does not depend on the energy.

Then for any $l > l_m$, one has

$$I_l \leqslant \frac{\int_0^\infty |\hat{j}_l(kr)| r^{a/2} |U_{\mathrm{T}}(r)| \,\mathrm{d}r}{[1 - X_a(l)]}. \tag{45}$$

Let us now come back to $|f_l|$. From (33), (35) and (45), we obtain

$$|f_l - f_l^{\mathrm{B}}| \equiv |f_l| \leqslant \frac{c'(c_1' l^{1/3} + c_1'')^{1-a}(2l+1)^{-a/2}[\int_0^\infty |\hat{j}_l(kr)| r^{a/2} |U_{\mathrm{T}}(r)| \,\mathrm{d}r]^2}{k^{2-a} X_a(l)}, \tag{46}$$

an inequality which becomes, with the help of the Schwarz inequality,

$$|f_l| \leqslant \frac{X_a(l)\{\int_0^\infty [\hat{j}_l(kr)]^2 |U_{\mathrm{T}}(r)| \,\mathrm{d}r\}}{k[1 - X_a(l)]}$$

$$|f_l| \leqslant \frac{X_a(l)}{1 - X_a(l)} |f_l^{\mathrm{B}'}|. \tag{47}$$

Also, for any given $\epsilon > 0$ the quantity $R_{l_1} = |f_l|/|f_l^{\mathrm{B}'}|$ is less than ϵ as soon as $l > l_0$, l_0 being greater than l_m and defined by

$$X_a(l_0) = \epsilon[1 - X_a(l_m + 1)]. \tag{48}$$

Now we show that $|f_l^{\mathrm{B}'}|$ itself can be bounded by a decreasing function of l.

5.2. *Study of* $|f_l^{\mathrm{B}'}|$

In reduced variables $|f_l^{\mathrm{B}'}|$ may be written

$$|f_l^{\mathrm{B}'}| = k \int_0^\infty [\hat{j}_l(kr)]^2 |\mathscr{V}_{\mathrm{T}}(kr)| \,\mathrm{d}r = \int_0^\infty [\hat{j}_l(x)]^2 |\mathscr{V}_{\mathrm{T}}(x)| \,\mathrm{d}x \tag{49}$$

and is seen to be independent of the chosen energy. As before, this is due to the fact that $V_{\mathrm{T}}(r)$ depends on the energy. Obviously, this has the consequence that, if we find a bound for $|f_l^{\mathrm{B}'}|$, this bound will only depend on l. We can write

$$|f_l^{\mathrm{B}'}| = \int_0^R [\hat{j}_l(x)]^2 |\mathscr{V}_{\mathrm{T}}(x)| \,\mathrm{d}x + \int_R^\infty [\hat{j}_l(x)]^2 |\mathscr{V}_{\mathrm{T}}(x)| \,\mathrm{d}x = f_l^1(R) + f_l^2(R). \tag{50}$$

Let us first study $f_l^2(R)$. For any given $\eta > 0$, one may find $R_i(\eta)$ such that, for any $x > R_i(\eta)$, one has (Bateman 1953 III)

$$\left|\mathrm{J}_{\nu_i}(c_i x) - \left(\frac{2}{\pi c_i x}\right)^{1/2} \cos\left(c_i x - \frac{\nu_i \pi}{2} - \frac{\pi}{4}\right)\right| < \frac{\eta}{(c_i x)^{1/2}} \left(\frac{2}{\pi}\right)^{1/2}$$

(indeed, the difference is a multiple of $[\sin(c_i x - \nu_i \pi/2 - \pi/4)]/(c_i x)^{3/2}$). Let us choose $R = \max_i\{R_i(\eta)\}$:

$$f_l^2(R) = \int_R^\infty \mathrm{d}x\ \mathrm{J}_{l+\frac{1}{2}}^2(x) \prod_i |(2/\pi c_i x)^{1/2} \cos(c_i x - \tfrac{1}{2}\nu_i \pi - \tfrac{1}{4}\pi)$$
$$+ [\mathrm{J}_{\nu_i}(c_i x) - (2/\pi c_i x)^{1/2} \cos(c_i x - \tfrac{1}{2}\nu_i \pi - \tfrac{1}{4}\pi)]| x^{\nu_0 + 2n}.$$

As the cos is always bounded by one, we have

$$|f_l^2(R)| \leq \prod_i [(2/\pi c_i)^{1/2}](1+\eta) \int_0^\infty \mathrm{d}x\ \mathrm{J}_{l+\frac{1}{2}}^2(x) x^{\nu_0 + 2n - 1/2}. \tag{51}$$

This last integral is convergent as soon as

$$2l + 2 > \tfrac{1}{2} - \nu_0 - 2n > 0. \tag{52}$$

It is easy to choose $l > l_2$ defined by $l_2 = -\frac{3}{2} - \nu_0 - 2n$. On the other hand, the inequality concerning ν_0 and n is not always fulfilled (compare with inequality (19)). We shall limit ourselves to the potentials satisfying condition (52). In this last case, the value of the integral appearing in equation (51) is known (Bateman 1953 V), and Stirling's formula allows one to write, for l large enough ($l > l_2'$),

$$|f_l^2(R)| < \prod_i [(2/\pi c_i)^{1/2}](1+\eta) c_2 l^{2n + \nu_0 - 1/2} \qquad c_2 > 0 \tag{53}$$

and as soon as l is greater than l_2, this last bound is a decreasing function of l.

Let us now come to $f_l^1(R)$. The function $|x^2 \mathscr{V}_\mathrm{T}(x)|$ is bounded near the origin (in order to apply scattering theory to $\mathscr{V}_\mathrm{T}(x)$). As $\mathscr{V}_\mathrm{T}(x)$ is a continuous function of x, one can write

$$|x^2 \mathscr{V}_\mathrm{T}(x)| \leq M \qquad \text{on the interval } [0, R]$$
$$|f_l^1(R)| \leq \tfrac{1}{2} M \pi \int_0^\infty \mathrm{J}_{l+\frac{1}{2}}^2(x) \frac{\mathrm{d}x}{x}.$$

For any given $\zeta > 0$, it is possible to find $l_2''(R, \zeta)$ such that, for any $l \geq l_2''$, the following inequality holds:

$$\begin{aligned} &\mathrm{J}_{l+\frac{1}{2}}(x) < \left(\frac{2}{\pi}\right)^{1/2} \frac{x^{l+1/2}}{(2l+1)!!}(1+\zeta) \qquad \text{for } x \in [0, R] \\ &|f_l^1(R)| \leq \frac{M}{[(2l+1)!!]^2}(1+\zeta)^2 \frac{R^{2l+1}}{(2l+1)}. \end{aligned} \tag{54}$$

Let us now choose $l \geq L = \max(l_2, l_2', l_2'')$. Then

$$|f_l^{\mathrm{B}'}| < c_2' l^{2n + \nu_0 - 1/2}(1+\eta) + c_2''(1+\zeta)^2 \frac{R^{2l+1}}{(2l+1)[(2l+1)!!]^2} \tag{55}$$

and, for any given $\zeta'>0$, one can find L' such that, for any $l \geqslant L'$, the following inequality holds:

$$c_2'' \frac{R^{2l+1}}{(2l+1)[(2l+1)!!]^2} \leqslant \zeta' c_2' l^{-1/2+\nu_0+2n}.$$

Then, for any $l \geqslant L_0 = \max\{L, L'\}$, we can write

$$|f_l^{B'}| < c_2' l^{-1/2+\nu_0+2n}[(1+\eta)+\zeta'(1+\zeta)^2] \tag{56}$$

and for any given $\epsilon'>0$, $|f_l^{B'}|$ is less than ϵ', as soon as $l \geqslant \max\{L_0, L_1\}$, L_1 being defined by

$$c_2' L_1^{-1/2+\nu_0+2n} = \epsilon'. \tag{57}$$

5.3. *Compatibility of the bounds for R_{l_1} and $|f_l^{B'}|$*

In the last two subsections, bounds for R_{l_1} and $|f_l^{B'}|$ have been independently derived. An interesting point is to determine whether or not the two categories of potentials for which § 5.1 and § 5.2 apply are compatible. In both cases, a condition on l was found: $l > l_0$ in § 5.1 and $l > \max\{L_0, L_1\}$ in § 5.2 (see the text for the definitions of l_0, L_0 and L_1). The compatibility of these conditions is obvious for l large enough.

The quantity R_{l_1} is bounded if a value of $a<1$ does exist such that the following inequalities hold:

$$\tfrac{2}{5} < a < m/2 - \nu_0 - 2n \tag{58}$$

$$\nu_m > -n - a/2. \tag{59}$$

One may easily be convinced that any potential answering to these conditions fulfils inequality (19). However, the converse is not true, and equations (58) and (59) are more restrictive than equation (19). In particular, equation (58) implies

$$\nu_0 < m/2 - 2n - \tfrac{2}{5}. \tag{60}$$

On the other hand, we have bounded $|f_l^{B'}|$, for l large enough, for the values of ν_0 such that

$$\nu_0 < \tfrac{1}{2} - 2n. \tag{61}$$

Except for $m=1$, this last condition is stronger than (60); however, the two conditions are never mutually exclusive.

To conclude, it is always possible to define a subclass amongst the class of transparent potentials defined by equation (18) and which obey condition (19). This subclass fulfils the inequalities

$$\nu_0 < \tfrac{1}{10} - 2n \qquad \text{for } m=1 \tag{62a}$$

$$\nu_0 < \tfrac{1}{2} - 2n \qquad \text{for } m>1 \tag{62b}$$

and ν_m must be such that it is possible to find a value of $a \in]\tfrac{2}{5}, 1[$ such that

$$\nu_m > -n - a/2. \tag{63}$$

For these potentials, R_{l_1} and $|f_l^{B'}|$ may at once be bounded for l large enough, and their bounds are found to be independent of the energy.

6. Conclusion

In this paper we have exhibited a relatively wide class of potentials which are either transparent or quasi-transparent at any energy in the Born approximation. All of them are energy dependent.

Their existence shows the lack of uniqueness of the solution of most inverse problems at fixed energy, only based upon the knowledge of the Born phase-shifts. Indeed, if a theoretical solution is sought for, the number of phase-shifts involved in the problem may be infinite, and a transparent potential may appear. If a numerical study is performed, the set of phase-shifts involved in the calculation is necessarily finite, and the solution may include one of the quasi-transparent potentials we have found. Unless an inner mechanism of the method prevents such transparent or quasi-transparent potentials from appearing—as is the case for the transparent potential in the Newton–Sabatier method—oscillations will generally be observed in the solution of the inverse problem.

This result does not contradict the result of Loeffel (1968): in the class of potentials which decrease asymptotically faster than any power of the variable, no transparent potential can be found.

We have not been able to obtain exact transparent or quasi-transparent potentials. However, the example of the Newton–Sabatier transparent potential, as well as the study of the bounds of the scattering amplitude of our transparent potentials, lead us to believe that the Born approximation is a 'good' approximation for large values of the angular momentum. As is well known, the large values of l correspond to the large values of the variable r. So, our conclusion is that, at least asymptotically, the solution of an exact inverse problem is generally non-unique, and that this lack of uniqueness is probably due to the existence of exact transparent potentials, the asymptotic decrease of which being one of those of the transparent potentials we have obtained in the Born approximation.

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Appendix

We want to study the integral

$$I_l = \int_0^\infty \mathrm{d}x\, x^{\nu_0+2n}[\mathrm{J}_{l+\frac{1}{2}}(x)]^2 \mathrm{J}_{\nu_1}(c_1x)\mathrm{J}_{\nu_2}(c_2x)\ldots\mathrm{J}_{\nu_m}(c_mx) \tag{A1}$$

with ν_0 defined by

$$\nu_0 = \nu_m - (\nu_1 + \ldots + \nu_{m-1}). \tag{A2}$$

We suppose that the parameters c_i fulfil condition (19). Then we know that, if c_m is such that

$$c_m > 2 + c_1 + c_2 + \ldots + c_{m-1}, \tag{A3}$$

I_l vanishes identically for any integer l. We shall use Lebesgue's theorem to show the continuity of I_l when (A3) holds and c_m goes to the limiting value

$$c_m = 2 + c_1 + c_2 + \ldots + c_{m-1}. \tag{A4}$$

Let us introduce the function $F(x, c_i)$ defined by

$$F(x, c_i) = |x^{\nu_0+2n}[\mathrm{J}_{l+\frac{1}{2}}(x)]^2 \mathrm{J}_{\nu_1}(c_1x)\mathrm{J}_{\nu_2}(c_2x) \ldots \mathrm{J}_{\nu_m}(c_mx)|.$$

For a given set of $c_1, c_2, \ldots, c_{m-1}$, $F(x, c_i)$ is a continuous function of c_m. Furthermore, it can be bounded independently of c_m by a summable function $G(x, c_1, c_2, \ldots, c_{m-1})$. This last function may be defined by cutting the integration domain of I_l into three regions:

(i) near the origin

$$G(x, c_1, \ldots, c_{m-1}) = c_1^{\nu_1} c_2^{\nu_2} \ldots c_{m-1}^{\nu_{m-1}} (c_0 x)^{\nu_m} x^{\nu_m+2n+2l+1}$$

where c_0 is the value of c_m which maximises $|\mathrm{J}_{\nu_m}(c_mx)|$ in this region;

(ii) on a finite interval

$$G(x, c_1, c_2, \ldots, c_{m-1}) = M_1|x^{\nu_0+2n}[\mathrm{J}_{l+\frac{1}{2}}(x)]^2 \mathrm{J}_{\nu_1}(c_1x) \ldots \mathrm{J}_{\nu_{m-1}}(c_{m-1}x)|$$

where $M_1 = \max_{c_m,x} |\mathrm{J}_{\nu_m}(c_mx)|$ on this interval;

(iii) in the asymptotic region common to the $(m+2)$ Bessel functions:

$$G(x, c_1, c_2, \ldots, c_{m-1}) = (c_1 \ldots c_{m-1})^{-1/2}(2 + c_1 + \ldots + c_{m-1})^{-1/2} x^{\nu_0+2n-m/2-1}.$$

In each of its three domains of definition, $G(x, c_1, c_2, \ldots, c_{m-1})$ is summable. So G is summable on the half-line $x > 0$.

Then the conditions for the applicability of Lebesgue's theorem are fulfilled, and we may include the limit (A4) in the domain of variations of the parameters c_i.

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